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G. Von Gehelen: PRODUCTION OF CHARGED INTERMEDIATE BO-
SONS IN HIGH-ENERGY NEUTRINO EXPERIMENTS.

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Production of Charged Intermediate Bosons in High-Energy Neutrino Experiments.

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Summary. — The total cross-section for the production of charged vector bosons by neutrinos in the electromagnetic field of a nucleus is calculated for different boson masses and nuclei. The spin summations and one integration are performed explicitly. For zero lepton mass near threshold the cross-section is found to obey a $\frac{7}{2}$ -power law. An explicit expression for the coherent part by the cross-section at high neutrino energies is given. For the incoherent contribution the Fermi motion of the nucleons in the target nucleus is taken into account.

1. — Introduction.

The Brookhaven and CERN neutrino experiments and especially the first results of the Brookhaven group (1) have shown the feasibility of high-energy experiments on weak interactions. There is evidence for the muon neutrino being different from the electron neutrino, but the question of the existence of intermediate vector bosons (2-9) is still open. But this question too may be solved in the near future.

Calculations for the production rate of the intermediate boson in various processes have been performed by many authors, the production by high-energy neutrinos has been calculated by LEE and YANG (2), LEE, MARKSTEIN

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(1) G. DANBY, J.-M. GAILLARD, K. GOULIANOS, L. M. LEDERMAN, N. MISTRY, M. SCHWARTZ and J. STEINBERGER: *Phys. Rev. Lett.*, **9**, 36 (1962).

(2) T. D. LEE and C. N. YANG: *Phys. Rev. Lett.*, **4**, 307 (1960); *Proc. of the 1960 Ann. Intern. Conf. on High-Energy Physics at Rochester*, p. 567, referred to as LY.

and YANG⁽³⁾, SOLOV'EV and TSUKERMAN⁽⁴⁾, and by VELTMAN⁽⁵⁾. The related process of the boson production by muons has been treated first by EBEL and WALKER⁽⁶⁾.

LY give the high-energy limit of the cross-section, SOLOV'EV and TSUKERMAN calculate the high-energy behaviour by means of the covariant Weizsäcker-Williams approximation of GRIBOV, KOLKUNOV, OKUN and SHEKHTER⁽⁷⁾. VELTMAN's work generalizes the technique of high-energy Coulomb wave functions⁽⁸⁾ to the case of spin one particles. Our work is similar to the calculations of LMY, but whereas LMY compute the cross-section by integrating each spin state separately by means of a fast electronic computer, it is the purpose of the present work to do all the spin summations and as many integrations as possible explicitly. So we get formulae which can be readily evaluated. Near threshold the incoherent contribution is found to obey a $\frac{7}{2}$ -power law. The velocity distribution of the target nucleons in the nucleus is taken into account by integrating over the Fermi sphere of the nucleus. For the coherent part of the cross-section at high-energies we give an explicit expression which goes beyond the Weizsäcker-Williams approximation.

2. - Summation over spins and polarization.

We want to calculate the production of vector mesons ("W-bosons") according to the two Feynman graphs of Fig. 1. The neutrino-lepton-W-vertex is assumed to contain a semi-weak interaction. The whole process shall take

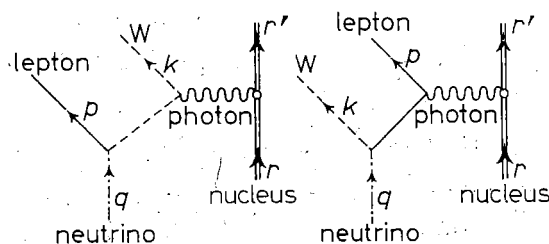


Fig. 1. - Feynman diagrams for W-production by neutrinos.

⁽³⁾ T. D. LEE, P. MARKSTEIN and C. N. YANG: *Phys. Rev. Lett.*, **7**, 429 (1961), referred to as LMY.

⁽⁴⁾ V. V. SOLOV'EV and I. S. TSUKERMAN: *Zhurn. Éksp. Teor. Fiz.*, **42**, 1252 (1962) English translation: *Sov. Phys. JETP*, **15**, 868 (1962).

⁽⁵⁾ M. VELTMAN: *Physica*, **29**, 161 (1963).

⁽⁶⁾ M. E. EBEL and W. D. WALKER: *Phys. Rev.*, **122**, 1639 (1961).

⁽⁷⁾ V. N. GRIBOV, V. A. KOLKUNOV, L. B. OKUN and V. M. SHEKHTER: *Zhurn. Éksp. Teor. Fiz.*, **41**, 1839 (1961), English translation: *Sov. Phys. JETP*, **14**, 1308 (1962).

⁽⁸⁾ H. A. BETHE and L. C. MAXIMON: *Phys. Rev.*, **93**, 768 (1954).

place in the electromagnetic field of a nucleus. Two different mechanisms are considered:

1) The photon interacts coherently with all the nucleons in the nucleus, i.e. it interacts with the Coulomb field of the nucleus. This process is important at high-energies, where the momentum transferred to the nucleus is predominantly small.

2) The photon interacts incoherently with the individual nucleons of the nucleus. This case is important near threshold, where only high momentum transfers are allowed.

We shall first calculate the cross-section for the neutrino incident on a free nucleon with anomalous magnetic moment. By summing over the Fermi distribution of the nucleons in the nucleus we get the incoherent contribution, whereas the coherent contribution is obtained assuming the nucleon to be infinitely heavy and replacing the nucleon form factor by the charge form factor of the nucleus.

For the W and its interactions we use the following terms in the Lagrangian density:

$$(1) \quad \left\{ \begin{aligned} & -\frac{1}{2} G_{\mu\nu}^* G_{\mu\nu} - M^2 U_{\mu}^* U_{\mu} + iG \bar{\psi}^{(l)} \gamma_{\mu} (1 + \gamma_5) \psi^{(v)} U_{\mu} + \\ & \qquad \qquad \qquad + \text{herm. conj. of the last term,} \\ & G_{\mu\nu} = \partial_{\mu} U_{\nu} - \partial_{\nu} U_{\mu}, \quad \partial_{\mu} = \partial/\partial x_{\mu} - ieA_{\mu}. \end{aligned} \right.$$

Here U_{μ} , $\psi^{(l)}$, $\psi^{(v)}$ are the field operators of the W, lepton, and neutrino, respectively. We use the metric $p^2 = \mathbf{p}^2 + p_4^2$, hermitian γ -matrices, and $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$. The coupling constant G is determined by the requirement that the boson W should transmit the weak interactions ⁽⁹⁾.

$$(2) \quad \sqrt{8} G_v = 4G^2/M^2,$$

where G_v is the vector coupling constant of μ -decay. M is the W-mass, m is the mass of the nucleon.

According to the Feynman-Dyson rules, the cross-section is given by

$$(3) \quad \sigma_{\text{tot}} = \frac{1}{(2\pi)^5} \frac{e^4 G^2 m^2}{|rq|} \frac{1}{2} \sum \int \left| \bar{w}(\mathbf{p}) \left\{ \frac{\gamma_e T_{e\tau\mu}}{(p-q)^2 + M^2} + \gamma_{\mu} \frac{i\gamma(k-q)}{(k-q)^2} \gamma_{\tau} \right\} \cdot \right. \\ \left. \cdot (1 + \gamma_5) u(\mathbf{q}) e_{\tau} \cdot \bar{v}(\mathbf{r}') \{ F_1(s^2) \gamma_{\mu} + (g/2m) F_2(s^2) \sigma_{\mu\nu} s_{\nu} \} v(\mathbf{r}) \right|^2 \\ \cdot s^{-4} \delta^4(q + s - p - k) \frac{d^3 p}{p_0} \frac{d^3 p'}{2k_0} \frac{d^3 r'}{r'_0}.$$

⁽⁹⁾ T. D. LEE and C. N. YANG: *Phys. Rev.*, **119**, 1410 (1960); **126**, 2239 (1962); **128**, 885 (1962); T. D. LEE: *Phys. Rev.*, **128**, 899 (1962).

The nomenclature of the respective momenta is obvious from Fig. 1, e_τ is the polarization vector of the W, $s = r - r'$, $Q = p - q$, and

$$(4) \quad T_{e\tau\mu} = -i(\delta_{e\sigma} + Q_e Q_\sigma / M^2)(-Q_\tau \delta_{\mu\sigma} + Q_\mu \delta_{\sigma\tau} - k_\mu \delta_{\sigma\tau} + k_\sigma \delta_{\mu\tau}).$$

The rest mass of the lepton has been neglected throughout. We conveniently split (2) into two parts, the contribution of the left-hand part of the Feynman diagrams in Fig. 1:

$$(5) \quad W_{\mu\nu} = \sum \left[\bar{w}(\mathbf{p}) \left\{ \gamma_e \frac{T_{e\tau\mu}}{M^2 - 2(pq)} + \gamma_\mu \frac{i\gamma(k-q)}{M^2 - 2(qk)} \gamma_\tau \right\} (1 + \gamma_5) u(\mathbf{q}) e_\tau \right] \cdot \left[\bar{w}(\mathbf{p}) \left\{ \gamma_\sigma \frac{T_{\sigma\zeta\nu}}{M^2 - 2(pq)} + \gamma_\nu \frac{i\gamma(k-q)}{M^2 - 2(qk)} \gamma_\zeta \right\} (1 + \gamma_5) u(\mathbf{q}) e_\zeta \right]^*,$$

and the contribution of the nucleon current:

$$(6) \quad S_{\mu\nu} = m^2 \sum [\bar{v}(\mathbf{r}') \{ F_1 \gamma_\mu + (g/2m) F_2 \sigma_{\mu e} s_e \} v(\mathbf{r})] \cdot [\bar{v}(\mathbf{r}') \{ F_1 \gamma_\nu + (g/2m) F_2 \sigma_{\nu\tau} s_\tau \} v(\mathbf{r})]^*.$$

Performing the summations over the spins and polarizations in $W_{\mu\nu}$, we get

$$(7) \quad W_{\mu\nu} = \frac{1}{4} \text{Tr} \left\{ -(\gamma p) \gamma_e \cdot 2(1 + \gamma_5) (\gamma q) \gamma_\sigma T_{e\tau\mu} T_{\sigma\zeta\nu} (M^2 - 2(pq))^{-2} + \right. \\ \left. + 2i(\gamma p) \gamma_e \cdot 2(1 + \gamma_5) (\gamma q) \gamma_\zeta \gamma(k-q) \gamma_\nu \frac{T_{e\tau\mu}}{(M^2 - 2(pq))(-M^2 - 2(qk))} + \right. \\ \left. + (\gamma p) \gamma_\mu \gamma(q-k) \gamma_\tau \cdot 2(1 + \gamma_5) (\gamma q) \gamma_\zeta \gamma(q-k) \gamma_\nu \frac{1}{(M^2 + 2(qk))^2} \left(\delta_{\zeta\tau} + \frac{k_\zeta k_\tau}{M^2} \right) \right\} = \\ = [(pq) \{ (p_\mu + q_\mu)(p_\nu + q_\nu) - 4Q_\mu k_\nu - M^2 \delta_{\mu\nu} \} + 2(pk) q_\mu k_\nu + 2(qk) p_\mu k_\nu - \\ - 2(pq)^2 \delta_{\mu\nu} + 2(pk)(qk)(2p_\mu q_\nu / M^2 - \delta_{\mu\nu}) + (pq)(kQ)^2 \delta_{\mu\nu} / M^2 - \\ - \{ 2(qk)^2 p_\mu p_\nu + 2(pk)^2 q_\mu q_\nu + 2k_\mu Q_\nu (pq)(kQ) \} / M^2] \frac{-2}{(M^2 - 2(pq))^2} + \\ + [(pq) \{ M^2 \delta_{\mu\nu} + 3Q_\mu k_\nu - 4p_\mu q_\nu \} + (pk)(3q_\mu q_\nu - p_\mu q_\nu) + \\ + (qk) \{ p_\mu q_\nu + p_\mu p_\nu - 4p_\mu k_\nu \} + 2(pq)^2 \delta_{\mu\nu} - (pq)(pk) \delta_{\mu\nu} + \\ + 2(qk)^2 \{ p_\mu q_\nu + p_\mu p_\nu \} / M^2 + 2(pq)(qk)(Qk) \delta_{\mu\nu} / M^2 - (pq)(qk)(2Q_\mu k_\nu / M^2 + \delta_{\mu\nu}) + \\ + 2(pk)(qk) \{ \delta_{\mu\nu} - p^\mu q^\nu / M^2 - q_\mu q_\nu / M^2 \}] \frac{-4}{(M^2 - 2(pq))(M^2 + 2(qk))} +$$

$$+ [(pq) M^2 \delta_{\mu\nu} + 8(qk)(p_\mu q_\nu - p_\mu k_\nu) - 2M^2 p_\mu q_\nu - 4(pq)(qk) \delta_{\mu\nu} + 4(pk)(qk) \delta_{\mu\nu} + 8(qk)^2 p_\mu q_\nu / M^2 - 4(pq)(qk)^2 \delta_{\mu\nu} / M^2] \frac{2}{(M^2 + 2(qk))^2}.$$

The expression (7) should be symmetrized in μ and ν , but for the following, the form given is sufficient.

$$(8) \quad S_{\mu\nu} = \frac{1}{4} \text{Tr} [(i\gamma r - m) \{i(F_1 + gF_2)\gamma_\mu - (g/2m)F_2(r_\mu + r'_\mu)\} \cdot (i\gamma r' - m) \{i(F_1 + gF_2)\gamma_\nu - (g/2m)F_2(r_\nu + r'_\nu)\}] = A \delta_{\mu\nu} + (B/4 - A/s^2) s_\mu s_\nu + Br_\mu r_\nu - (B/2)(r_\mu s_\nu + r_\nu s_\mu),$$

where

$$(9) \quad A = (s^2/2)(F_1 + gF_2)^2, \quad B = 2F_1^2 + (s^2/(2m^2))g^2F_2^2.$$

So (3) becomes

$$(10) \quad \sigma_{\text{tot}} = \frac{1}{(2\pi)^3} \frac{\alpha^2 G^2}{|(rq)|} \int W_{\mu\nu} S_{\mu\nu} \frac{1}{s^4} \delta^4(q + r - p - k - r') \frac{d^3p}{p_0} \frac{d^3k}{k_0} \frac{d^3r'}{r'_0},$$

with $\alpha=1/137$, and $W_{\mu\nu}$, $S_{\mu\nu}$ given by (7) and (8).

3. - Integration over the final momenta.

We now use the fact (*) that $W_{\mu\nu}$ does not depend on r and r' separately, but only on the combination $s = r - r'$. Introducing a δ -function $\delta^4(s + q - p - k)$, we get

$$(11) \quad \sigma_{\text{tot}} = \frac{1}{(2\pi)^2} \frac{\alpha^2 G^2}{|(rq)|} \int \frac{d^4s}{s^4} \frac{d^3r'}{r'_0} \delta^4(s - r + r') T_{\mu\nu} S_{\mu\nu}.$$

In (11) we have put

$$(12) \quad T_{\mu\nu} = \frac{1}{2\pi} \int \frac{d^3p}{p_0} \frac{d^3k}{k_0} W_{\mu\nu} \delta^4(s + q - p - k).$$

The integrations over d^3r' and ds_0 are used to eliminate the δ -function $\delta^4(s - r + r')$. In the special system $\mathbf{r} = 0$ one is left with

$$(13) \quad \sigma_{\text{tot}} = \frac{1}{2\pi} \frac{\alpha^2 G^2}{|(rq)|} \int \frac{d|\mathbf{s}|^2 d(\mathbf{s}\mathbf{q})}{2q_0 r'_0 s^4} S_{\mu\nu} T_{\mu\nu}.$$

(*) See ref. (7). Eqs (16) and (46)-(48) have already been obtained by A. POMPEI: *Thesis* (Cagliari, 1960), unpublished.

Observing that in the system $\mathbf{r} = 0$:

$$(14) \quad \begin{cases} \mathbf{s}^2 = s^2 + s^4/(4m^2), \\ (\mathbf{s}\mathbf{q}) = (sq) - s^2(rq)/(2m^2), \\ q_0 r'_0 = -(qr)(1 + s^2/2m^2), \end{cases}$$

one rewrites (13) in invariant form:

$$(15) \quad \sigma_{\text{tot}} = \frac{\alpha^2 G^2}{2(2\pi)(qr)^2} \int dx dy S_{\mu\nu} T_{\mu\nu} / x^2,$$

where $x = s^2$, $y = (sq)$.

On the other hand it follows from gauge and Lorentz invariance that

$$(16) \quad T_{\mu\nu} = \frac{1}{2}(T_2 - T_1)\delta_{\mu\nu} - T_1 s_\mu s_\nu / s^2 + \\ + \frac{1}{2}(T_2 - 3T_1)\{(s^2 q_\mu q_\nu / (sq)^2 - (s_\mu q_\nu + s_\nu q_\mu) / (sq)\},$$

where T_1 and T_2 are invariants of the tensor $T_{\mu\nu}$:

$$(17) \quad T_1 = - (s^2 / (sq)^2) T_{\mu\nu} q_\mu q_\nu, \quad T_2 = T_{\mu\mu}.$$

From (12), T_1 and T_2 are integrals over d^3p and d^3k which can be evaluated analytically. Explicit expressions for T_1 and T_2 are given in the Appendix.

Using the results (8), (9) for $S_{\mu\nu}$ and (16) for $T_{\mu\nu}$, we get for the main factor of the integrand in (15)

$$(18) \quad S_{\mu\nu} T_{\mu\nu} = (F_1^2 + xg^2 F_2^2 / (4m^2)) \cdot \\ \cdot (T_1 \{m^2 - x/2 + 3x(rq)/y - 3x(rq)^2/y^2\} + T_2 \{m^2 - x(rq)/y + x(rq)^2/y^2\}) + \\ + F_1^2 T_2 (x/2 - 2m^2) + xg F_1 F_2 T_2.$$

Equations (15), (18) and the formulae for T_1 and T_2 given in the Appendix are our result for the cross-section for W-production on a free proton. In general, the two integrations over x and y have to be performed numerically. Our remaining task is the determination of the limits of integration in (15). First, we look for the range of $x = s^2$.

We define $t = p + k$, the momentum of the particles produced in our process. Obviously $t^2 \leq -M^2$. We express x in terms of the variables t^2 and $\cos\varphi = \hat{q}\hat{t}$, and the constant quantities (qr) and m :

$$(19) \quad x = (q - t)^2 = \\ = (2(qr)^2 - t^2(qr) + m^2 t^2 - 2(qr) \cos\varphi \{((qr) - t^2/2)^2 + m^2 t^2\}^{1/2}) / (m^2 - 2(qr)).$$

The extremal values for x are reached for $\cos \varphi = \pm 1$ and for the largest possible value of t^2 , i.e. for $t^2 = -M^2$.

The limits for $y = (sq)$ for fixed x and fixed (qr) follow from

$$(20) \quad (q + s)^2 = 2y + x = t^2.$$

The largest value of y is reached when t^2 assumes its largest value, $-M^2$. The lowest possible value of t^2 for fixed (qr) and x has to be calculated from (19). Solving (19) for t^2 , one finds that the extremal values of t^2 are reached for $\cos^2 \varphi = 1$ and are given by

$$(21) \quad t_{\text{extr}} = (1 - (qr)/m^2)x \pm ((qr)/m^2)(x^2 + 4xm^2)^{\frac{1}{2}}.$$

The plus sign is to be excluded, because it leads to a value of t^2 larger than $-M^2$. So we have finally

$$(22) \quad x_{\text{min}}^{\text{max}} = \frac{2(qr)^2 + M^2(qr) - M^2m^2 \mp 2(qr)\{(qr) + M^2/2\}^2 - M^2m^2\}^{\frac{1}{2}}}{m^2 - 2(qr)},$$

and

$$(23) \quad \begin{cases} y_{\text{max}} = -(x + M^2)/2, \\ y_{\text{min}} = \{x(qr)/(2m^2)\}(\sqrt{1 + 4m^2/x} - 1). \end{cases}$$

4. - The incoherent contribution to the total cross-sections.

In Fig. 2 we give the results of our numerical integration of (15), (18) for three different choices of the W mass: $M = 0.6, 1.2,$ and 1.8 GeV. For the nucleon form factors we used the formula given by LITTAUER, SCHOPPER and WILSON (10):

$$(24) \quad \begin{cases} F_1^p(x) = 0.16 + \frac{0.55t_{R1}}{x + t_{R1}} + \frac{0.29t_{R2}}{x + t_{R2}}, \\ g_p F_2^p(x) = -0.41 + \frac{2.113t_{R1}}{x + t_{R1}} + \frac{0.09t_{R2}}{x + t_{R2}}, \end{cases}$$

where $t_{R1} = 0.346 m^2, t_{R2} = 0.184 m^2$. The accuracy of our numerical integrations is $\sim 25\%$.

(10) R. M. LITTAUER, H. SCHOPPER and R. R. WILSON: *Phys. Rev. Lett.*, **7**, 141 (1961).

Using (24) for all x , the contribution of the region $x > 35$ fermi⁻², where the form factor is not well known experimentally, is $< 2\%$ for $M = 0.6$ GeV, $< 6\%$ for $M = 1.2$ GeV and $< 20\%$ for $M = 1.8$ GeV. The magnetic moment and interference terms contribute $(20 \div 35)\%$ for $M = 1.2$ GeV and $q_0 < 6$ GeV.

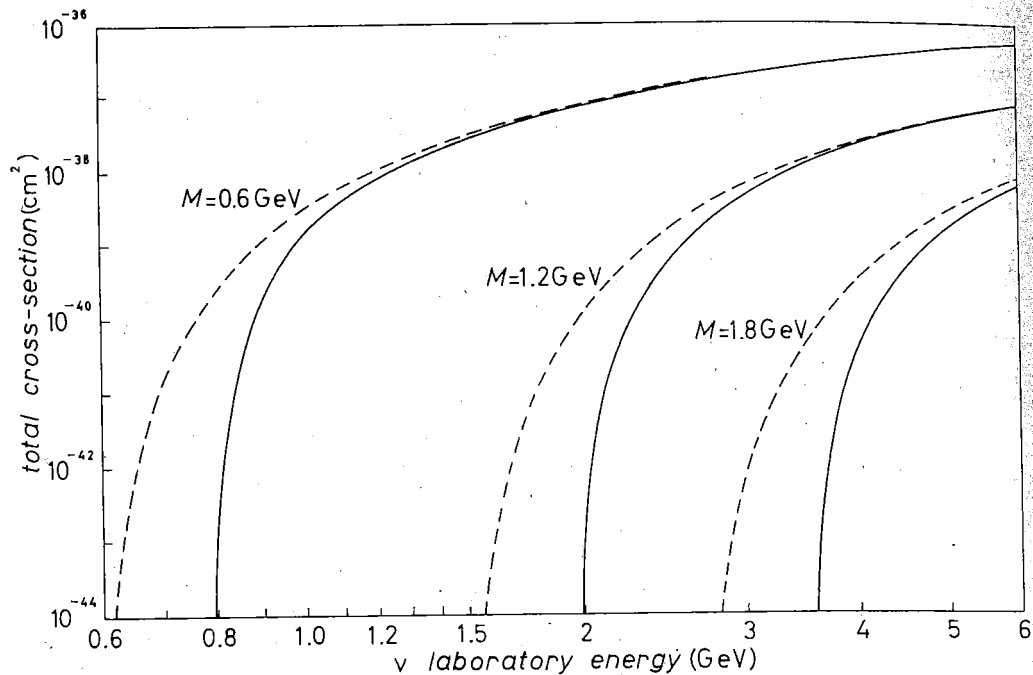


Fig. 2. - Total cross-section for a neutrino incident on a proton, $\nu + p \rightarrow W + l + p$, for three choices of the boson mass M . Full lines: proton taken initially at rest. Broken lines: contribution per proton, when the initial Fermi distribution of the protons in a nucleus is taken into account.

The threshold neutrino-energy is

$$(25) \quad q_{\text{th}} = M(1 + M/(2m)) .$$

It has to be recalled however, that we have always neglected the rest mass of the lepton. The most conspicuous alteration of our results due to a finite lepton mass is the displacement of the threshold. Therefore, in our formulae one should always replace M by $M + \mu$, where μ is the lepton mass. In (22) and also in (23) this already gives the correct result, whereas the change in the matrix element is more complicated, see for instance eq. (A.3) and (A.9) of the Appendix.

Another displacement of the threshold is due to the Fermi motion of the nucleons in the target nucleus. Calling T the Fermi energy of the nucleus,

($T \approx 0.04$ GeV), the threshold is lowered to

$$(26) \quad q_{\text{th Fermi}} = \frac{M(1 + M/(2m))}{1 + T/m + \sqrt{2T/m + (T/m)^2}},$$

i.e., it is reduced by a factor of ≈ 1.33 . If we assume that the momentum transfer to the nucleon in consideration is always so large that the nucleon leaves the nucleus, we can disregard the Pauli principle. This will be true for low neutrino energies. Then the main effect of the Fermi motion in the nucleus is purely kinematical: we have to average over the velocity distribution of the initial nucleons. The total cross-section depends only on the invariant quantity (qr). So we get the cross-section for an initial proton moving with momentum r and under an angle θ with respect to the incident neutrino, if in Fig. 2 we take as abscissa

$$q[\sqrt{1 + (r/m)^2} - (r/m) \cos \theta],$$

instead of q_0 . Introducing $R = r/m$, the integration over the Fermi sphere reads

$$(27) \quad \sigma_{\text{Fermi}}(q_0) = \frac{3N}{2\varrho^3} \int_{R_{\text{min}}}^{\varrho} dR R^2 \int_{-1}^{(\cos \theta)_{\text{max}}} d(\cos \theta) \sigma[q_0(\sqrt{1 + R^2} - R \cos \theta)].$$

Here

$$(28) \quad \varrho = ((T/m)^2 + 2T/m)^{\frac{1}{2}}$$

is the Fermi momentum in units of the nucleon mass, N is the number of nucleons in the nucleus.

The region of integration in (27) is shown in Fig. 3. For $q < q_{\text{th}}$, the region is bounded by a curve of type I, shrinking to zero for $q = q_{\text{th Fermi}}$. For $q = q_{\text{th}}$ we have curve II. If $q_{\text{th}} < q < q_{\text{th}}/(\sqrt{1 + \varrho^2} - \varrho)$, there is still a region of large backward nucleon mo-

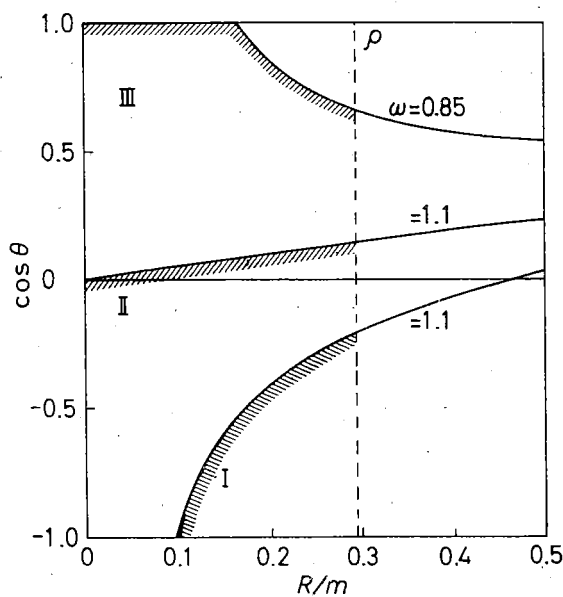


Fig. 3. - Boundaries for the integration over the Fermi sphere of a nucleus. $\omega = q_{\text{th}}/q$, where q_{th} is the W-production threshold for a neutrino incident on a free nucleon. $\varrho \cdot m$ is the Fermi radius in momentum space.

menta inside the Fermi sphere which give center-of-mass energies below production threshold. Explicitly the limits in (27) may be written:

$$(29) \quad (\cos \theta)_{\max} = \text{Max} \{ \text{Min} (+1, -\omega/R + \sqrt{1 + 1/R^2}), -1 \},$$

$$(30) \quad R_{\min} = \text{Min} \left\{ \text{Max} \left(0, \frac{\omega^2 - 1}{2\omega} \right), \varrho \right\},$$

where

$$(31) \quad \omega = q_{\text{th}}/q_0.$$

The integration in (27) are easily performed numerically, using a polynomial fit to the data of Fig. 2. The results are given in the same Fig. 2 (broken curves).

5. - Threshold approximation for the incoherent case.

In this section the integration over x and y will be performed explicitly for neutrino energies near threshold. Instead of (qr) we introduce a quantity ϱ defined by

$$(32) \quad (qr) = -Mm \left(1 + \frac{M}{2m} \right) - \frac{M+m}{M} \varrho.$$

If the initial nucleon is at rest, this is equivalent to

$$(33) \quad q_0 = q_{\text{th}} + \frac{M+m}{Mm} \varrho.$$

In lowest order of ϱ the limits (22) of the x -integration become

$$(34) \quad x_{1,2} = \frac{M^2 m}{M+m} \mp \frac{M+2m}{(M+m)^{\frac{3}{2}}} M \sqrt{2m\varrho}.$$

From (34) it is obvious that at threshold the range of the x -integration reduces to the point

$$(35) \quad x_{\text{th}} = \frac{M^2 m}{M+m}.$$

We now define two new variables ξ and η by

$$(36) \quad x = x_{\text{th}} + \xi, \quad y = -(x + M^2)/2 + \eta.$$

In terms of ξ the limits of integration (34), (23) become

$$(37) \quad \xi_{1,2} = \mp \alpha \sqrt{\varrho}, \quad \eta_2 = 0, \quad \eta_1 = \xi^2/\alpha^2 - \varrho,$$

where we have introduced

$$(38) \quad \alpha = \frac{M + 2m}{(M + m)^{\frac{3}{2}}} M \sqrt{2m}.$$

We see that η is always small of the order ϱ . Accordingly we may expand the integrand in powers of η . The expansion of T_1 and T_2 is given in the Appendix. Apart from the factor η^2 in T_1 and T_2 , all other terms are slowly varying functions of ξ and η in the region of integration. So we are allowed to evaluate them for $\xi = 0$ and $\eta = 0$. For the total cross-section we get an expression of the following form:

$$(39) \quad \sigma_{\text{tot}} = h \int_{-\alpha\sqrt{\varrho}}^{\alpha\sqrt{\varrho}} d\xi \int_{\xi^2/\alpha^2 - \varrho}^0 d\eta \eta^2,$$

where ϱ and α are given by (33) and (38), while h depends only on the choice of the boson mass M :

$$(40) \quad h = \frac{G_v}{4\pi\sqrt{2}(137)^2} \left\{ \left(F_1^2 + \frac{x_{\text{th}}}{4m^2} g^2 F_2^2 \right) \frac{2R_1}{x_{\text{th}}} + (F_1 + gF_2)^2 \frac{R_2 M^4}{2(qr)_{\text{th}}^2 x_{\text{th}}} \right\},$$

R_1 and R_2 are given in the Appendix, eq. (A.8). The integrations in (39) are readily performed, resulting in

$$(41) \quad \sigma_{\text{tot}} = h \frac{32\alpha}{105} \varrho^{\frac{7}{2}} = 1.44 \cdot 10^{-38} \text{ cm}^2 \left(\frac{q - q_{\text{th}}}{1 \text{ GeV}} \right)^{\frac{7}{2}} x^3 \left(\frac{m}{M} \right)^{\frac{3}{2}} \left\{ \left(F_1^2 + \frac{xM^2}{4m^2} g^2 F_2^2 \right) \cdot \frac{4m(M + 2m)}{M^2} (x + 1)(x + 2)^2 + (F_1 + gF_2)^2 (x^4 + 2x^3 - 3x^2 - 8x + 8) \right\},$$

where, in the last lines, x has to be taken at threshold and in units of M^2 , i.e. $x = m/(M + m)$. We write (41) in the form

$$(42) \quad \sigma_{\text{tot}} = A \left(\frac{q - q_{\text{th}}}{1 \text{ GeV}} \right)^{\frac{7}{2}},$$

where A is a function of the boson mass only. In Table I we give some values for A . In order to show, up to which energies (42) represents a good approximation of formula (15), (18), in Fig. 4 we have plotted the results of our exact

TABLE I. - Dependence of A (defined by eq. (42)) on the boson mass M .

| M (GeV) | A (10^{-40} cm 2) | M (GeV) | A (10^{-40} cm 2) |
|-----------|----------------------------|-----------|----------------------------|
| 0.5 | 21400 | 1.2 | 117 |
| 0.6 | 7770 | 1.8 | 8.39 |
| 0.9 | 710 | 3.0 | 0.342 |

numerical integration of (15), (18), now as a function of $\log(q - q_{th})$. One sees that the simple power law (42) turns out to be approximately true up to

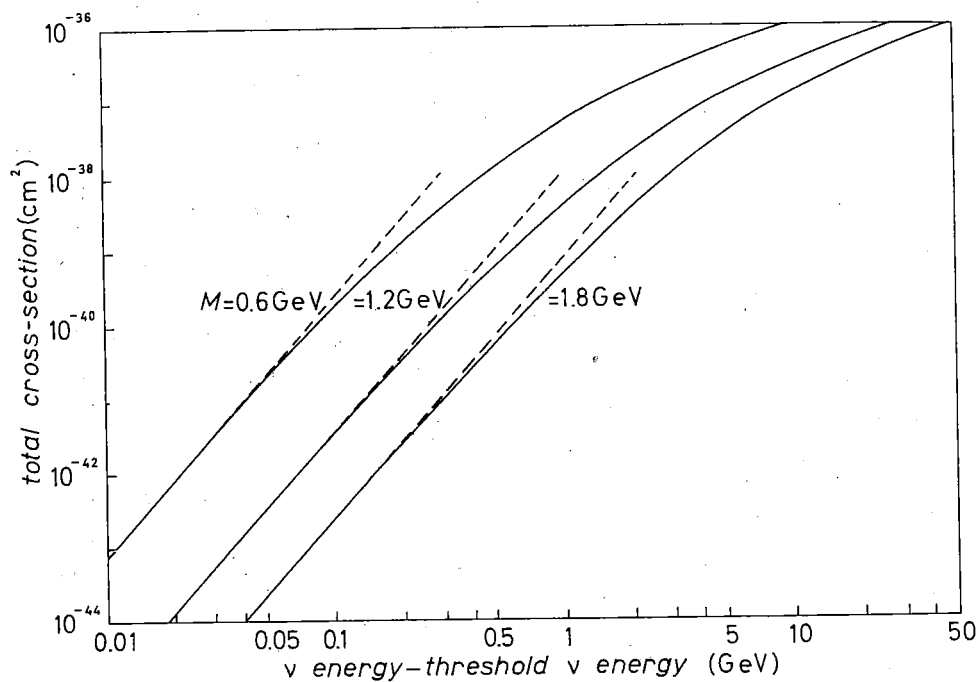


Fig. 4. - Total cross-section for $\nu + p \rightarrow W + l + p$, plotted against $q - q_{th}$. Full lines: results of the exact integration of eq. (15), (18). Broken lines: threshold approximation eq. (41).

energies (0.2 ÷ 0.5) GeV above threshold. Of course, here the influence of the lepton mass, which has been neglected in our calculation, should be taken into account. This may be done in a straightforward manner by calculating also the terms neglected in (7).

If the cross-section obeys a power law as in (42), one may even perform the integration over the Fermi sphere of the initial nucleon states analytically. The result is for our case (42)

$$(43) \quad \sigma_{\text{Fermi}} = \frac{3Aq^{\frac{1}{2}}}{2q^3} \left\{ (F_+ + G_+) - (F_- + G_-) \theta \left(1 - \varrho + \frac{\varrho^2}{2} - \omega \right) \right\},$$

where

$$(44) \quad \left\{ \begin{aligned} F_{\pm} &= \frac{4}{99} X_{\pm}^{11} - \frac{1 \pm \varrho}{20} \left\{ \frac{4}{9} X_{\pm}^9 + dX_{\pm}^7 + \frac{7}{3} d^2 X_{\pm}^5 + \frac{35}{6} d^3 X_{\pm}^3 + \frac{35}{2} d^4 X_{\pm} \right\}, \\ G_{\pm} &= \mp \frac{7\sqrt{2}}{4} d^5 \log \left| \frac{X_{\pm} + (\varrho \pm 1)/\sqrt{2}}{\sqrt{|\omega - \frac{1}{2}|}} \right|, \\ d &= \frac{1 - 2\omega}{4}, \quad X_{\pm} = (\varrho^2/2 \pm \varrho + 1 - \omega)^{\frac{1}{2}}, \end{aligned} \right.$$

$m \cdot \varrho$ is the radius of the Fermi sphere, $\omega = q_{\text{th}}/q$, and

$$\theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

6. - The total cross-section in the coherent case.

If we want to neglect completely that the nucleus can recoil as a whole, we get the results for the coherent case by performing the limit $m \rightarrow \infty$ in (15), (18), (22), and (23). Clearly, we now use the laboratory system where $(rq) = -mq_0$. T_1 and T_2 do not contain m , so we get instead of (18)

$$(45) \quad \lim_{m \rightarrow \infty} S_{\mu\nu} T_{\mu\nu} = mF_1^2 \{ T_1(1 - 3xq_0^2/y^2) + T_2(-1 + xq_0^2/y^2) \}.$$

The limits of integration reduce to

$$(46) \quad x_{\min}^{\max} = (q_0 \pm \sqrt{q_0^2 - M^2})^2,$$

$$(47) \quad y_{\max} = -(x + M^2)/2, \quad y_{\min} = -q_0\sqrt{x}.$$

For the form factor F_1 we now have to use $Z \cdot F(x)$, where $F(x)$ is the Fourier transform of the nuclear charge distribution. So, finally we get

$$(48) \quad \sigma_{\text{coh}} = \frac{Z^2 G_{\nu}}{8\sqrt{2}\pi^2(137)^2} \int \frac{dx}{(q_0 x)^2} F^2(x) \int dy \left\{ T_1 \left(1 - \frac{3xq_0^2}{y^2} \right) + T_2 \left(-1 + \frac{xq_0^2}{y^2} \right) \right\}.$$

The nuclear form factor $F(x)$ is unit for $x \ll R^{-2}$, where R is the radius of the nucleus. For $x > R^{-2}$ the form factor depends sensitively on the exact shape of the nuclear charge distribution, which is not well known. So the results for the coherent cross-section become dubious for low neutrino energies, where $x_{\min} > R^{-2}$. Approximating (46) by $x_{\min} = (M^2/2q_0)^2$, this means $q_0 < RM^2/2$. For copper and $M = 1.2$ GeV, one has $R \cdot M \approx 35$, which means that the coherent cross-section for $q_0 < 21$ GeV is determined only by the unknown tail of the nuclear form factor. Of course for these neutrino energies the allowed

momentum transfers are so large, that the cross-section will be to a large degree incoherent.

In our numerical evaluation of (48), we used the trapezoidal model for the nuclear charge distribution. The corresponding form factor has been given by HERMAN and HOFSTADTER⁽¹¹⁾:

$$(49) \quad F(x) = \frac{12}{1-\eta^4} \frac{1}{\zeta^4} \{ \eta \zeta \sin(\eta \zeta) - \zeta \sin \zeta + 2 \cos(\eta \zeta) - 2 \cos \zeta \},$$

with

$$(50) \quad \begin{cases} \zeta = R\sqrt{x}, & R = \frac{1}{m} (5.09 A^{\frac{1}{3}} + 7.13), \\ \eta = \frac{5.09 A^{\frac{1}{3}} - 7.13}{5.09 A^{\frac{1}{3}} + 7.13}. \end{cases}$$

In the region of q_0 , where $x_2 \approx R^{-2}$, we find that σ does not increase monotonically with q_0 , but shows deviations of the order of a factor 2 from an average curve. This is due to the oscillation of $F(x)$ for $x > R^{-2}$. One may regard σ as uncertain at least to this order. Our results are plotted in Fig. 5 and 6.

It should be noted that the integration over y is independent of the nucleus chosen. If we measure q_0 in units of M , it is also independent of the mass of the boson. This fact reduces considerably the amount of numerical work to be done. Once the integration over y has been performed for the relevant values of x , for each nucleus and each value of the boson mass only a single numerical integration is needed in order to get the total cross-section.

We have seen that $x \ll M^2$, for the region where the order of magnitude of $F(x)$ can be deduced from existing experiments. So we may expand the integrand of (48) for all cases of interest. Then it is possible to carry out the integration over y analytically. We find (with $a = q_0 \sqrt{x}/M^2$)

$$(51) \quad \frac{1}{M^2} \int_{-a}^{-(x+M^2)/2} dy \left\{ T_1 \left(1 - \frac{3M^4 a^2}{y^2} \right) + T_2 \left(-1 + \frac{M^4 a^2}{y^2} \right) \right\} =$$

$$= -13 - \frac{5}{24} - \frac{2\pi^2}{3} + \left(11 + \frac{13}{18} \right) a^2 + \frac{35}{3} a + \frac{20}{9a} + 4\mathcal{L}_2 \left(\frac{1}{2a} \right) +$$

$$+ (6 + a^2) \log^2(2a) + \left\{ -\frac{35a^2}{6} - 14a - \frac{1}{8} + \frac{8}{3a} - 4 \log(x/M^2) \right\} \log(2a) +$$

$$+ \left(\frac{16}{3} a^2 - 8a + \frac{4}{3a} \right) \log \frac{M^2(2a-1)}{x},$$

\mathcal{L}_2 is the dilogarithm $\mathcal{L}_2(z) = \sum_{n=1}^{\infty} z^n/n^2$ for $|z| < 1$.

(11) R. HERMAN and R. HOFSTADTER: *High-Energy Scattering Tables* (Stanford, 1960).

Fig. 5. - Coherent contribution to the total cross-section for $\nu + {}^{63}_{29}\text{Cu} \rightarrow W + l + {}^{63}_{29}\text{Cu}$. Full lines: complete formula (48). Broken lines: high-energy approximation eq. (52). The deformation of the curves is due to the use of an oscillating nuclear form factor.

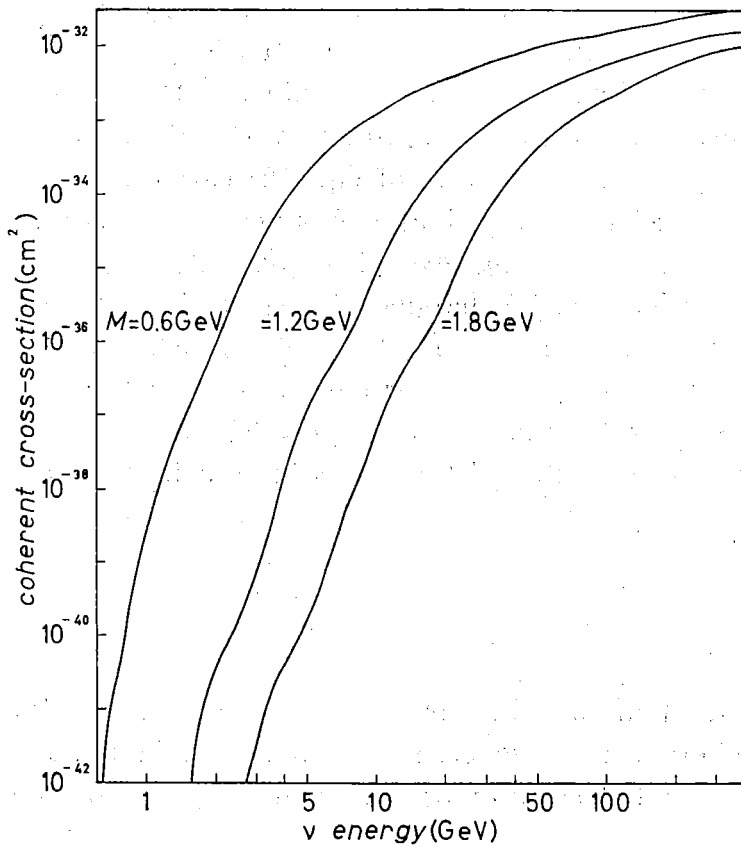
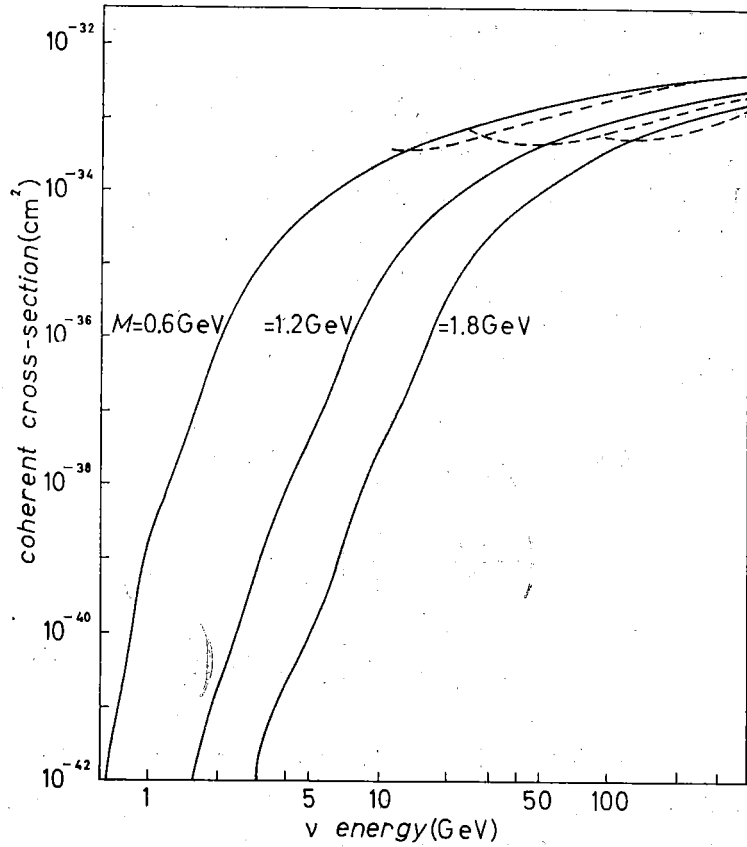


Fig. 6. - Coherent contribution to the total cross-section for $\nu + {}^{207}_{82}\text{Pb} \rightarrow W + l + {}^{207}_{82}\text{Pb}$.

Now this has only to be folded with the square of the form factor divided by x^2 in order to get the coherent cross-section. For very-high energies the main contribution comes from the region $x \ll R^{-2}$, where $F(x) \approx 1$. Putting $F(x) \equiv 1$ and using R^{-2} for the upper limit of the integration (in the exponential model we have $R = \bar{R}/\sqrt{12}$, where \bar{R} is the radius of the nucleus ⁽³⁾), we are able to perform also the integration over x . The resulting formula may be expanded in powers of q_0 without introducing new errors, as has been checked by comparison with the exact numerical integration of (48). The result is (with $\alpha = 2q_0/RM^2$)

$$(52) \quad \sigma_{\text{coh}} = \frac{Z^2 G_v}{4\sqrt{2} \pi (137)^2} \left\{ \frac{2}{3} \log^3 \alpha + \frac{61}{6} \log^2 \alpha + \left(-\frac{31}{3} + \frac{88}{\alpha} + \frac{32}{3} \log(RM)^2 \right) \log \alpha + \left(-\frac{152}{9} + \frac{32}{\alpha} \right) \log(RM)^2 - \frac{12}{\alpha} - \frac{40}{9} \pi^2 - \frac{133}{108} + 32 \left(\frac{1}{3} + \frac{1}{4 \cdot 2^2} + \frac{1}{5 \cdot 3^2} + \dots \right) \right\} + O(q^{-2} \log^2 q).$$

In the highest term of (52) one recognizes the extreme relativistic approximation of LEE and YANG ⁽²⁾. It is interesting that even at $q_0 = 2000$ GeV (for Cu and $M = 1.2$ GeV) this term amounts to only 15% of the whole expression (52), due to large numerical factors in the other terms.

A similar explicit calculation has been done by SOLOV'EV and TSUKERMAN ⁽⁴⁾. These authors take the lepton mass different from zero, but they use a covariant Weizsäcker-Williams approximation ⁽⁷⁾. In our notation, this approximation corresponds to neglecting T_1 and to approximate T_2 by its value for $x = 0$. This however, gives only the leading term of eq. (52) correctly. Their result $-\frac{1}{2}$ for the coefficient of $\log^2 \alpha$ (in our notation) is obtained, if in eq. (A.11) we would be allowed to neglect everything except $-y/2 - 2y \log(-2y)$. The contribution of $(4/y + 4) \log((1 + 2y)/x)$ happens to cancel in the order $\log^2 \alpha$, but there is a contribution from $2/y^2 \log((1 + 2y)/x)$, which may be regarded to be due to very small values of x , with y close to $-\frac{1}{2}$. The corresponding change in the high-energy formula is quite appreciable.

Our formula (52) reproduces the results of the exact integration down to $q_0 = RM^2$ within 40%, as is shown in Fig. 5.

7. - Conclusions.

The production of charged W-mesons in the electromagnetic field of a nucleus is in general due to coherent and incoherent contributions. In addition the nucleus may be left in an excited state or even additional particles may be produced.

At very-high energies the process is expected to proceed mostly through the coherent mechanism. Our explicit calculation in Section 6 shows that the Weizsäcker-Williams approximation, even in the covariant form, is not sufficient for the derivation of the high-energy cross-section. The dependence of the total cross-section for the partial process $\nu + \gamma \rightarrow W + l$ on the photon « mass », which is neglected in the W.-W.-approximation, enters already in the term proportional to $\log^2(2q_0/RM^2)$. Only the term proportional to $\log^3(2q_0/RM^2)$ which is numerically unimportant up to neutrino energies of more than 10^3 GeV is given correctly by the W.-W.-method. Instead our explicit formula (52) can be used down to about 50 GeV.

For the region between 5 and 20 GeV ⁽¹²⁾ it is difficult to make a reliable prediction of the cross-section, because the coherent part is still very important but it depends sensitively on the unknown tail of the nuclear form factor.

For neutrino energies below 5 GeV the incoherent part dominates, but nuclear excitation and additional particle production may play an important role. For these energies the Pauli principle should be taken into account.

Only very near threshold, where experiments are actually done, the momentum transfer to the nucleus is always larger than the Fermi momentum of the nucleus, so that we may neglect the effect of the Pauli principle. As has been shown in Section 4, it is a simple matter to account for the Fermi motion in the initial nucleus. As in the high-energy case, also at low energies a closed expression can be given for the total cross-section. In Section 5 we found that near threshold a $\frac{7}{2}$ -power law holds if the lepton mass is taken to be zero. Our explicit formula (41) gives the cross-section near threshold as a function of the W mass in the approximation of neglecting the muon mass. Work to include a finite muon mass into the threshold formula is in progress.

* * *

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⁽¹²⁾ For $M \approx 1.2$ GeV. For lower values of the W mass the relevant neutrino energies are correspondingly lower.

APPENDIX

In this appendix we calculate the quantities T_1 and T_2 , which are defined as

$$(A.1) \quad \left\{ \begin{array}{l} T_1 = -\frac{s^2}{2\pi(sq)^2} \int W_{\mu\nu} q_\mu q_\nu \delta^4(s+q-p-k) \frac{d^3p}{p_0} \frac{d^3k}{k_0} \\ \text{and} \\ T_2 = \frac{1}{2\pi} \int W_{\mu\mu} \delta^4(s+q-p-k) \frac{d^3p}{p_0} \frac{d^3k}{k_0} \end{array} \right.$$

The integrations are most easily performed in the system $\mathbf{p} + \mathbf{k} = \mathbf{s} + \mathbf{q} = 0$. Only the integration over (pq) is nontrivial. For T_2 we get

$$(A.2) \quad T_2 = \int_{-|p|q_0}^{+|p|q_0} W_{\mu\mu} \frac{d(pq)}{q_0(p_0+k_0)},$$

and an analogous expression for T_1 . (A.2) may be rewritten in invariant form:

$$T_2 = -\frac{1}{y} \int_{y(1+M^2/(x+2y))}^0 W_{\mu\mu} d(pq),$$

where we have put $x = s^2$, $y = (sq)$. If one does not want to neglect the muon mass μ , the limits of the integration over (pq) are to be substituted by

$$(A.3) \quad (pq)_{\min}^{\max} = \frac{y}{2(x+2y)} (z \mp \sqrt{z^2 + 4\mu^2(x+2y)}),$$

with $z = x + 2y + M^2 - \mu^2$.

Now $W_{\mu\nu} q_\mu q_\nu$ and $W_{\mu\mu}$ are to be calculated from (7), using

$$(A.4) \quad (pk) = (M^2 + s^2)/2 + y, \quad (qk) = -(pq) + y.$$

The integration over (pq) can be performed analytically, resulting in

$$(A.5) \quad T_1 = \frac{x+2y+M^2}{M^2x-2xy-4y^2} \left(x - \frac{4M^2x}{y} - \frac{x^2}{2y} + \frac{2M^4x}{y^2} + \frac{x^2M^2}{4y^2} \right) - \\ - \frac{x}{2M^2} + \frac{4M^2}{y} + \frac{x}{2y} - \frac{x^2}{4M^2y} + \frac{5M^2x}{2y^2} + \frac{2M^4}{y^2} + \frac{x^2}{2y^2} - \frac{M^2x}{4y(x+2y)} + \\ + \left(-\frac{M^6x}{y^4} - \frac{3M^2x^2}{8y^3} + \frac{x^2}{2y^2} + \frac{7M^2x}{4y^2} \right) \log \frac{M^2(x+2y)}{M^2x-2xy-4y^2} + \\ + \left(-\frac{2M^2x}{y^2} + \frac{M^6x}{y^4} \right) \log \frac{(M^2+2y)(x+2y)}{M^2x},$$

$$\begin{aligned}
 (A.6) \quad T_2 = & \frac{2x}{M^2} + \frac{3y}{2M^2} + \frac{2x+y}{x+2y} - 4 \frac{x+2y+M^2}{M^2+2y} - \frac{M^2y}{2(x+2y)^2} + \\
 & + \frac{x+2y+M^2}{M^2x-2xy-4y^2} \left(\frac{4y^2}{M^2} + \frac{2xy}{M^2} - x + 8M^2 - \frac{x^2}{2M^2} \right) + \\
 & + \left(\frac{x+2y}{M^2} - 3 - \frac{4M^2}{y} + \frac{x^2}{4M^2y} - \frac{5x}{2y} + \frac{2M^2x-2M^4}{y^2} \right) \log \frac{M^2(x+2y)}{M^2x-2xy-4y^2} + \\
 & + \left(4 + \frac{4M^2}{y} + \frac{2M^4-2M^2x}{y^2} \right) \log \frac{(M^2+2y)(x+2y)}{M^2x}.
 \end{aligned}$$

Both T_1 and T_2 vanish for $y = -(x + M^2)/2$. In Section 5 we need the expansion of T_1 and T_2 in the neighbourhood of $y = -(x + M^2)/2$. Putting $y = -(x + M^2)/2 + \eta$, we find in lowest order of η

$$(A.7) \quad T_1 = R_1 \eta^2, \quad T_2 = R_2 \eta^2,$$

with

$$(A.8) \quad \begin{cases} R_1 = -4(x + M^2)(x + 2M^2)^2 / (M^8x), \\ R_2 = (x + M^2)(x^4 + 2M^2x^3 - 3M^4x^2 - 8M^6x + 8M^8) / (M^{10}x^2). \end{cases}$$

If the muon mass μ is not neglected, the logarithms in (A.5), (A.6) read

$$(A.9) \quad \log \frac{x+2y+n^2-w}{x+2y+n^2+w} \quad \text{and} \quad \log \frac{x+2y-n^2+w}{x+2y-n^2-w},$$

with

$$w = \sqrt{(x+2y+M^2-\mu^2)^2 + 4\mu^2(x+2y)} \quad \text{and} \quad n^2 = (M^2 - \mu^2)(1 + x/y).$$

If x is small with respect to M^2 , in our region of integration (46), (47), the following approximate formulae hold:

$$(A.10) \quad T_1 = 4/y + 2/y^2,$$

$$\begin{aligned}
 (A.11) \quad T_2 = & -\frac{y}{2} - \frac{9}{2} - \frac{4x}{1+2y} - \frac{33}{8y} - \frac{2}{y^2} + \\
 & + \left(-2y + 3 + \frac{4}{y} + \frac{2}{y^2} \right) \log(-2y) + \left(4 + \frac{4}{y} + \frac{2}{y^2} \right) \log \frac{2y(1+2y)}{x}.
 \end{aligned}$$

In the last two formulae, M^2 has been taken to be unity.

RIASSUNTO (*)

Si calcola la sezione d'urto totale per la produzione di bosoni vettoriali carichi da neutrini nel campo elettromagnetico di un nucleo, per differenti masse bosoniche e diversi nuclei. Si eseguono esplicitamente le sommatorie degli spin ed una integrazione. Si trova che, per massa leptonica nulla presso la soglia, la sezione d'urto obbedisce ad una legge esponenziale in $\frac{1}{2}$. Si dà una espressione esplicita per la parte coerente della sezione d'urto alle alte energie del neutrino. Per il contributo incoerente si tien conto del moto di Fermi dei nucleoni nel nucleo del bersaglio.

(*) Traduzione a cura della Redazione.